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Period Doubling for Trapezoid Function Iteration: Metric Theory*

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Iterations of a one-parameter family $F(\lambda, x) = \lambda f(x)$ of endomorphisms of $[0, 2]$ having the form of a trapezoid $f(x) = x/e$ for $x \in [0, e]$, $f(x) = 1$ for $x \in (e, 2 - e)$ and $f(x) = (2 - x)/e$ for $x \in [2 - e, 2]$, are investigated. Here $\lambda \in [1, 2]$ and $e \in (0, 1)$. Let λ_n be the smallest value of $\lambda > 1$ for which $x = 1$ is a periodic point of period 2^n. It is proved that for $e < 0.99$, $\lambda_n - \lambda_{n-1} \simeq k(\lambda_\infty/e)^{-2^n}$, where k is some constant and $\lambda_\infty = \lim_{n \to \infty} \lambda_n$. The same conclusion probably holds for any $e < 1$. This behavior is substantially different from that found by Feigenbaum and others for the case where $f(x)$ assumes its maximum value for a unique x. Numerical investigations are reported for functions related to the trapezoid function.

1. INTRODUCTION

1.1. As has been known for over a decade, there is a large class of one-parameter maps of the interval whose periodic orbits under iteration exhibit a certain "structural" regularity which is universal for the class. The situation is most clearly set forth in the paper of Metropolis *et al.* [10], whose notation and terminology we adopt here. To summarize: let a transformation on (0, 1) be given by

$$T_\lambda(x) = \lambda f(x), \tag{1.1}$$

where λ is a real, positive parameter such that $T_\lambda(x)$ is into [0, 1]. Necessary conditions on $f(x)$ for the universal property to hold are not known, but in [10] the following conditions are asserted to be sufficient:

(a) $f(x)$ is continuous, piecewise C^1 on [0, 1], strictly positive on (0, 1), with $f(0) = f(1) = 0$.

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(b) The maximum $f_{\max} = 1$ of f is assumed either at a single point or over a single interval.

(c) For a unique x^* such that $f(x^*) = 1$, $f'(x^*)$ exists.

(d) There exists λ_0 such that for $\lambda_0 < \lambda < 1$, $\lambda f(x)$ has only two fixed points, the origin and $x_f(\lambda)$, say, both of which are repellent. That is to say, if

$$T_\lambda^k(x) = \underbrace{T_\lambda(T_\lambda \ldots (T_\lambda}_{k}(x)\ldots),$$

then

$$\lim_{k\to\infty} T_\lambda^k(y) \neq 0 \text{ or } x_f(\lambda)$$

for y sufficiently close to 0 or to $x_f(x)$.

(e) There exists an interval N which strictly includes the maximum point or interval, and on this N, $f(x)$ is concave. For every $x \notin N$, the piecewise derivative of $f(x)$ is greater than 1 in absolute value. To the left or right of N, f is strictly increasing or decreasing, respectively.

For convenience, we take the single maximum (or the center of the flat top) to lie at $x = 1/2$. For each k, the equation

$$T_\lambda^k(1/2) = 1/2 \tag{1.2}$$

has a number of solutions λ, each of which corresponds to a unique period of length k, one of whose points is $x = 1/2$. These periods are distinguished from each other by their "patterns," words of $k - 1$ letters on the alphabet (R, L). R indicates that an iterate lies to the right of $x = 1/2$, L that it lies to the left. Thus the (unique) 3-period (for any transformation in the class) is $1/2 \to R \to L \to 1/2$, written RL. The countably infinite set of all periods (for all k) can be uniquely ordered on increasing λ; the actual values of λ depend of course on $f(x)$. The resulting sequence of patterns and their ordering is universal for the class of T_λ's. We follow Derrida *et al.* [3] in referring to this set of patterns as the MSS sequence.

1.2. An important phenomenon connected with this sequence is what is now called "period doubling," referred to in [10] as "the existence of harmonics." Briefly, associated with any pattern P of length k (i.e., $k - 1$ letters), there is a set of patterns of length $k\ 2^n$, of well defined structure, called "harmonics." To each such harmonic there corresponds a solution of (1.2); the sequence $\{\lambda_n\}$ of these converges as $n \to \infty$ to some λ_∞. Furthermore, there are no solutions λ of (1.2) for *any* k such $\lambda_n < \lambda < \lambda_{n+1}$. It was

Feigenbaum [5] who first observed that the ratio

$$\delta_n = \frac{\lambda_n - \lambda_{n-1}}{\lambda_{n+1} - \lambda_n} \tag{1.3}$$

seems to converge to a value δ which is the same for all functions in some allowed class which have the same degree in the neighborhood of the maximum $x = 1/2$. Write

$$f(x) \cong 1 - (1 + a)(1/2 - x)^{1+\epsilon}, \tag{1.4}$$

(a and ϵ small and positive) which holds in a neighborhood of $x = 1/2$. Then for $\epsilon = 1$ Feigenbaum calculated

$$\delta_n \to \delta = 4.6692016\ldots \quad , \tag{1.5}$$

an observation that has been checked by others (including the present authors).

Feigenbaum's result is an example of "metric" universality (as contrasted with structural universality). Collet *et al.* ([1] or [2]) have shown the universality of the limit $\delta_n \to \delta(\epsilon)$ in a suitable class of functions for small ϵ and a, but at this writing no results for $\epsilon = 1$ have been published. See, however, Lanford [8]). These authors also proved another Feigenbaum conjecture [1], namely that δ is the unique eigenvalue greater than unity of the derivative of a certain "scaling" operator.

On the structural side, the universality of the MSS sequence needs demonstration. The sketch of a proof given in [10] contains a few lacunae; we think, however, these can be removed.

1.3. The conditions on $f(x)$ given in paragraph 1.1 assume only continuous piecewise C^1 functions. (In contrast, Guckenheimer [6] establishes a certain kind of universality of maps only for C^3 functions). If one enlarges the class of functions so as to include

$$\begin{aligned} l(x) &= 2x, && 0 \le x < 1/2, \\ &= 2(1 - x), && 1/2 \le x \le 1, \end{aligned}$$

one finds that the iterates of $l(x)$ omit harmonics as well as some other composite patterns. See [4] for further details.

Flat top functions, however, seem to have normal structural properties. The metric property of Feigenbaum is another matter. In this paper we consider the trapezoid function

$$\begin{aligned} f(x) &= \frac{1}{e}x, & 0 \le x \le e, \\ &= 1, & e < x \le 1-e, \\ &= \frac{1}{e}(1-x), & 1-e < x \le 1, \end{aligned} \tag{1.6}$$

with $0 < e < 1/2$. As we show in this paper, the first harmonic sequence

$$R,\ RLR,\ RLR^3LR, \ldots .$$

has the property that, for some constant k_1, as n increases:

$$\lambda_n - \lambda_{n-1} \cong k_1 (\lambda_\infty / e)^{-2^n}, \tag{1.7}$$

with $\lim \lambda_n = \lambda_\infty$. A more precise statement is given in Theorem 1. This is in contrast to Feigenbaum's quadratic case, where

$$\lambda_n - \lambda_{n-1} \cong k_2 (4.6692 \ldots)^{-n} \tag{1.8}$$

for some constant k_2. The numerical evidence suggests that (1.7) also holds (for the trapezoid function) for the 3-period harmonic sequence

$$RL,\ RL^2RL,\ RL^2RLR^2L^2RL, \ldots .$$

Further experimentation indicates that (1.7) also holds for several other flat top functions. We conjecture that (1.7) holds for all MSS patterns for all flat tops.

If (1.8) is regarded as geometric convergence, then (1.7) could be regarded as supergeometric convergence.

There are six reasons for the interest in trapezoidal functions and the asymptotic behavior of their harmonics.

1. The trapezoids can be regarded as the limit as $\epsilon \to \infty$ of a smooth $f(x)$ satisfying (1.4). See Section 9 for further discussion.
2. Trapezoid functions have corners and thus fail to satisfy smoothness requirements which are often used as hypotheses in discussions of functional iteration, as for example, in Guckenheimer [6].
3. It is desirable to find a more elementary proof for Feigenbaum's result (1.5), at least in the case of $f(x) = 4x(1-x)$. It has not yet been possible to carry the elementary proof given in this paper for the trapezoid over to the function $4x(1-x)$.
4. Any additions to the catalog of asymptotic behaviors for various endomorphisms of the unit interval are welcome.
5. Flat top functions occur in control and communication theory.

6. The theoretical and numerical evidence given in this paper indicates that any endomorphism $f(x)$ of $[0, 1]$ which satisfies:

(a) $f(0) = f(1) = 0$,

(b) there exists e with $0 < e < 1/2$ so that for $x < e, f'(x) > 0$ and for $x > 1 - e, f'(x) < 0$,

(c) $f(x) = 1$ for $e < x < 1 - e$,

satisfies the conclusion of Theorem 1. This implies that the relation (1.8) is unstable with respect to small changes in $f(x)$ near the maximum value.

2. Statement of the Main Theorem

In order to avoid factors of $1/2$, we shall find it convenient to work with the interval $[0, 2]$ rather than $[0, 1]$. Our trapezoid function then takes the form

$$\begin{aligned} g(x, e) &= \frac{1}{e}, && 0 \le x \le e, \\ &= 1, && e \le x \le 2 - e, \\ &= \frac{1}{e}(2 - x), && 2 - e \le x \le 2, \end{aligned} \tag{2.1}$$

with $0 < e < 1$. Instead of T, we shall denote the corresponding map by H:

$$H(x, e, \lambda) = \lambda g(x, e), \qquad 1 < \lambda < 2. \tag{2.2}$$

The notations for iterates will be:

$$H_1 = H,\ H_2 = H \circ H, \ldots,\ H_p = H \circ H \circ \ldots H \circ H \qquad (p \text{ factors}),$$

where $\circ$ denotes functional composition.

Let λ_1^e be the unique root of

$$H_2(1, e, \lambda) = 1 \tag{2.3}$$

for $\lambda > 2 - e$. We need the following lemma.

Lemma 2.1. *Suppose numbers $\lambda_1^e < \lambda_2^e < \cdots < \lambda_n^e$ are given and that λ_k^e is the smallest root exceeding λ_{k-1}^e of $H_k(1, e, \lambda) - 1$. Then there exists a root λ^* of $H_{n+1}(1, e, \lambda) - 1$ which is the smallest root $> \lambda_n^e$, and $\lambda^* < 2$.*

The proof of Lemma 2.1 is given in Section 3. Put $\lambda_{n+1}^e \equiv \lambda^*$. With this definition of λ_n^e, we have the following theorem.

THEOREM 1. *For* $e \leq 0.99$,

$$\lim_{n\to\infty} \frac{\log\{(\lambda_{n+1}^e - \lambda_n^e)/(\lambda_{n+2}^e - \lambda_{n+1}^e)\}}{\log\{(\lambda_n^e - \lambda_{n-1}^e)/(\lambda_{n+1}^e - \lambda_n^e)\}} = 2. \tag{2.4}$$

Remark. The upper bound on e is a practical matter; it could be raised significantly with sufficient effort. The theorem is almost certainly true for all $e < 1$. Of course, at $e = 1$, all $\lambda_n^1 = 1$.

3. PROOF OF THEOREM 1. PART I: PRELIMINARY MACHINERY

As shown in [10], the harmonic of a pattern P is the pattern $P\Gamma P$, where $\Gamma = L$ if P contains an odd number of R's, and $\Gamma = R$ otherwise. This enables us to write down the iterated map H_{2^j} corresponding to any harmonic of the trapezoid, using Eq. (2.1) for $g(x, e)$. Here it is convenient to introduce the variable

$$z = \lambda/e. \tag{3.1}$$

The first few harmonics lead to the expressions

$$\begin{aligned}
H_2(1,e,\lambda) &= -ez^2 + 2z,\\
H_4(1,e,\lambda) &= ez^4 - 2z^3 + 2z,\\
H_8(1,e,\lambda) &= -ez^8 + 2z^7 - 2z^5 + 2z^4 - 2z^3 + 2z,\\
H_{16}(1,e,\lambda) &= ez^{16} - 2z^{15} + 2z^{13} - 2z^{12} + 2z^{11} - 2z^9 + 2z^7 - 2z^5 + 2z^4\\
&\quad -2z^3 + 2z,
\end{aligned}$$

and so forth (there is a simple algorithm for translating a given pattern into the corresponding polynomial). The general expression for kth harmonic is necessarily less explicit:

$$H_{2^k}(1,e,\lambda) = (-1)^k ez^{2^k} + 2\sum_{n=1}^{2^k-1} a_n^{2^k} z^n, \qquad k \geq 1, \tag{3.2}$$

where

$$a_n^{2^k} = \pm 1, 0. \tag{3.3}$$

As in the examples given above, the $a_n^{2^k} \neq 0$ alternate in sign; in addition,

$$a_n^{2^k} = 0 \Rightarrow a_{n\pm 1}^{2^k} \neq 0. \tag{3.4}$$

Let

$$f_k(z, e) = H_{2^k}(1, e, \lambda) - 1. \tag{3.5}$$

It is easy to show that

$$f_k(z, e) = -z^{2^{k-1}} f_{k-1}(z, e) + f_{k-1}(z, 1). \tag{3.6}$$

Define $z_n = \lambda_n^e / e$. In Table I we give a list of these roots to two decimal places for a range of e values and $n = 1, 2, 3, 4$.

LEMMA 3.1.

$$f_n(z_n, 1) = (-1)^n z_n^{2^n}(1 - e). \tag{3.7}$$

This follows immediately from Eq. (3.6).

LEMMA 3.2.

$$f_n(z_{n-1}, e) = (-1)^{n-1} z_{n-1}^{2^{n-1}}(1 - e). \tag{3.8}$$

Noting that $f_{n-1}(z_{n-1}, e) = 0$ by definition, we see that the lemma is an immediate consequence of (3.6) and (3.7).

TABLE I

e	z_1	z_2	z_3	z_4
0.001	1999.50	2000.00	2000.00	2000.00
0.01	199.50	200.00	200.00	200.00
0.05	39.49	39.98	39.98	39.98
0.1	19.49	19.95	19.95	19.95
0.3	6.12	6.52	6.53	6.53
0.5	3.41	3.75	3.77	3.77
0.7	2.21	2.48	2.52	2.52
0.9	1.46	1.65	1.71	1.71
0.95	1.29	1.44	1.50	1.51
0.99	1.11	1.21	1.26	1.28
0.999	1.03	1.13	1.14	1.15
0.9999	1.01	1.04	1.07	1.08

LEMMA 3.3. *There exist positive c_1 and c_2, independent of n (but possibly dependent on e) such that*

$$c_1 \leq \frac{1}{z_{n-1}^{2^n}} |f_n'(z_{n-1}, e)| \leq c_2 \tag{3.9}$$

for $n \geq 4$ and $e \leq 0.99$.

Proof. It is convenient to carry out the proof for even n; the proof for odd n differs only in a few minor details. Taking the derivative of Eq. (3.5), we have

$$\begin{aligned} f_n'(z, e) = e2^n z^{2^n-1} &- 2(2^n - 1)z^{2^n-2} + 2(2^n - 3)z^{2^n-4} \\ &-2(2^n - 4)z^{2^n-5} + \dots -2\left[2^n - (2^n - 2^{n-1} - 1)\right]z^{2^{n-1}} \\ &+2\left[2^n - (2^n - 2^{n-1} + 1)\right]z^{2^{n-1}-2} - \dots \\ &-2(2^n - (2^n - 3))z^2 + 2(2^n - (2^n - 1)). \end{aligned} \tag{3.10}$$

We now split $f_n'(z, e)$ into four parts as follows.

$$\begin{aligned} g_n^1(z, e) = e2^n z^{2^n-1} &- 2\cdot 2^n z^{2^n-2} + 2\cdot 2^n z^{2^n-4} - 2\cdot 2^n z^{2^n-5} \\ &+2\cdot 2^n z^{2^n-6} - 2\cdot 2^n z^{2^n-8} + \dots -2\cdot 2^n z^{2^{n-1}}; \end{aligned} \tag{3.11}$$

$$g_n^2(z, e) = 2^n\left\{2z^{2^{n-1}-2} - 2z^{2^{n-1}-4} + \dots -2z^2 + 2\right\}; \tag{3.12}$$

$$h_n^1(z) = 2z^{2^n-21}P_{19}(z), \tag{3.13}$$

with $P_{19}(z)$ the polynomial

$$\begin{aligned} P_{19}(z) = z^{19} &- 3z^{17} + 4z^{16} - 5z^{15} + 7z^{13} - 9z^{11} + 11z^9 \\ &-12z^8 + 13z^7 - 15z^5 + 16z^4 - 17z^3 + 19z - 20; \end{aligned} \tag{3.14}$$

finally,

$$\begin{aligned} h_n^2(z) = 2\big\{&(21z^2 - 23)z^{2^n-24} + (25z^2 - 27)z^{2^n-28} + (28z - 29)z^{2^n-30} \\ &+ \dots + \left[(2^n - 3)z^2 - (2^n - 1)\right]\big\}. \end{aligned} \tag{3.15}$$

Note that the general term of $h_n^2(z)$ is of the form

$$2\cdot(az^{b-a} - b)z^{2^n-b-1}, \qquad \text{where } b - a = 1 \text{ or } 2.$$

It can be verified that

$$f_n'(z,e) = g_n^1(z,e) + g_n^2(z,e) + h_n^1(z) + h_n^2(z). \tag{3.16}$$

Let us rewrite the functions g_n^1 and g_n^2:

$$2^{-n}g_n^1(z,e) = -z^{2^{n-1}-1}\left\{-ez^{2^{n-1}} + 2z^{2^{n-1}} - 2z^{2^{n-1}-3} + \dots +2z - 1 + 1\right\},$$

$$\begin{aligned}2^{-n}g_n^2(z,e) &= z^{-1}\left\{ez^{2^{n-1}} - ez^{2^{n-1}} + 2z^{2^{n-1}-1} - \dots -2z^3 + 2z - 1 + 1\right\}\\ &= z^{-1}\left\{ez^{2^{n-1}} - f_{n-1}(z,e)\right\},\end{aligned}$$

so that

$$g_n^1(z_{n-1},e) + g_n^2(z_{n-1},e) = 2^n\left\{z_{n-1}^{2^{n-1}-1}(e-1) + z_{n-1}^{-1}\right\}.$$

Now there is a $k > 1$ such that $1 < k \le z_{n-1} < 2$, and therefore

$$\lim_{n\to\infty} \frac{g_n^1(z_{n-1},e) + g_n^2(z_{n-1},e)}{z_{n-1}^{2^n}} = 0 \tag{3.17}$$

By an application of Sturm's sequence (as evaluated on the MACSYMA computer system [9]), we have found that the polynomial $P_{19}(z)$ of (3.14) has a unique positive root $z = 1.27\dots$, so clearly there exist positive c_3, c_4 such that

$$c_4 \ge P_{19}(z) \ge c_3 \qquad \text{for } 1.28 \le z \le 2. \tag{3.18}$$

Therefore, by (3.13),

$$c_6 \ge \frac{h_n^1(z)}{z^{2^n}} \ge c_5 > 0 \tag{3.19}$$

for some c_5, c_6, with $1.28 \le z < 2$.

It is not hard to show that $z_n(e) > z_n(e')$ for $e < e'$. It follows that from Table I that for $e \le .99$ and $n > 4$, $z_n(e) > 1.28$. Use of (3.15), (3.16), (3.17), (3.18), and (3.19) completes the proof of Lemma 3.3.

Finally, we can now give a proof of Lemma 2.1. Suppose that n is even. Then from (3.8),

$$f_n(z_{n-1},e) < 0.$$

Also,

$$f_n(k, e) = ek^{2^n} - 2k^{2^n-1} + 2k^{2^n-3} - 2k^{2^n-4} + \dots + 2k - 1$$
$$= k^{2^n-1}(ek-2) + 2\cdot k^{2^n-4}(k-1) + \dots + 2k - 1,$$

which is positive for $k \geq 2/e$. Hence

$$z_n < 2/e$$

or

$$\lambda_n^e < 2.$$

Similar remarks can be made for n odd. This completes the proof of Lemma 2.1.

4. Proof of Theorem I. Part II: Estimate of Remainder in Newton's Formula

Since $f_n(z, e)$ is a polynomial, it has a Taylor expansion and we can write

$$0 = f_n(z_n, e) = f_n(z_{n-1}, e) + (z_n - z_{n-1})f_n'(z_{n-1}, e) + \frac{(z_n - z_{n-1})^2}{2} f_n''(\xi_1, e), \tag{4.1}$$

where

$$z_{n-1} < \xi_1 < z_n. \tag{4.2}$$

Solve (4.1) for $z_n - z_{n-1}$, choosing n to be even so that $f_n' < 0$. An expansion of the square root in the expression for $z_n - z_{n-1}$ yields

$$z_n - z_{n-1} = -\frac{f_n(z_{n-1}, e)}{f_n'(z_{n-1}, e)} - \frac{1}{2}(1 - \xi_2)^{-3/2} \frac{f_n^2(z_{n-1}, e) f_n''(\xi_1, e)}{[f_n'(z_{n-1}, e)]^3}, \tag{4.3}$$

where

$$0 < \xi_2 < -\frac{2f_n(z_{n-1}) f_n''(\xi_1)}{[f'(z_{n-1})]^2}, \tag{4.4}$$

(or with inequalities reversed in (4.4), depending on the sign of the last expression in (4.4)). The expansion of the square root is valid provided

$$\left|2\frac{f_n(z_{n-1})f_n''(\xi_1)}{[f_n'(z_{n-1})]^2}\right| < 1. \tag{4.5}$$

We shall show subsequently that (4.5) holds for sufficiently large n.

5. Proof of Theorem 1. Part III: Other Estimates and Conclusion

In the following, we assume that $1 < k_1 < z \leq 2$ and that $1 < k_1 < z_{n-1} \leq 2$.

For n even, we have on differentiating (3.10):

$$\begin{aligned} f_n''(z, e) &= e2^n(2^n - 1)z^{2^n-2} - 2(2^n - 1)(2^n - 2)z^{2^n-3} \\ &\quad + 2(2^n - 3)(2^n - 4)z^{2^n-5} + \cdots - 12z. \end{aligned} \tag{5.1}$$

Consequently there exists a constant k_2 so that

$$|f_n''(z, e)| \leq k_2 2^n z^{2^n-2}. \tag{5.2}$$

Using Lemmas 3.2 and 3.3 and (5.2), we have for some $k_3 > 0$,

$$\left|\frac{2f_n(z_{n-1}, e)f_n''(\xi_1, e)}{[f_n'(z_{n-1}, e)]^2}\right| \leq \frac{k_3 z_{n-1}^{2^{n-1}} 2^n \xi_1^{2^n-2}}{z_{n-1}^{2^{n+1}}} \tag{5.3}$$

$$= \frac{k_3 2^n}{z_{n-1}^{3\cdot 2^{n-1} - (2^n-2)\log \xi_1/\log z_{n-1}}}. \tag{5.4}$$

Thus ξ_2 in (4.4) can be made arbitrarily small in magnitude for large n.

We also need the estimate with some constant $k_4 > 0$:

$$\left|\frac{f_n^2(z_{n-1}, e)f_n''(\xi_1, e)}{[f_n'(z_{n-1}, e)]^3}\right| \leq \frac{k_4 z_{n-1}^{2^n} 2^n \xi_1^{2^n-2}}{z_{n-1}^{3\cdot 2^n}} = k_4 \frac{2^n}{z_{n-1}^{2^n - (2^n-2)\log \xi_1/\log z_{n-1}}}. \tag{5.5}$$

The next estimate needed follows from Lemmas 3.2 and 3.3. There exist k_5

and k_6 so that

$$\frac{k_5}{z_{n-1}^{2^{n-1}}} < \left| \frac{f_n(z_{n-1}, e)}{f_n'(z_{n-1}, e)} \right| < \frac{k_6}{z_{n-1}^{2^{n-1}}} \tag{5.6}$$

for sufficiently large n.

There follows from (4.3), (5.4), (5.5), and (5.6) the following.

LEMMA 5.1. *For $e \leq 0.99$ and n sufficiently large, there exist constants k_7 and k_8 independent of n so that*

$$\frac{k_7}{z_{n-1}^{2^{n-1}}} < z_n - z_{n-1} < \frac{k_8}{z_{n-1}^{2^{n-1}}} \tag{5.7}$$

or

$$z_n - z_{n-1} = \frac{w_n}{z_{n-1}^{2^{n-1}}}, \tag{5.8}$$

where $k_8 > w_n > k_7$.

We can now compute the quantity in (2.4), using (3.1) and (5.8):

$$\frac{\log\{(\lambda_{n+1}^e - \lambda_n^e)/(\lambda_{n+2}^e - \lambda_{n+1}^e)\}}{\log\{(\lambda_n^e - \lambda_{n-1}^e)/(\lambda_{n+1}^e - \lambda_n^e)\}}$$

$$= \frac{2^n\left(2\dfrac{\log z_{n+1}}{\log z_n} - 1\right)\log z_n + \log(w_{n+1}/w_{n+2})}{2^{n-1}\left(2\dfrac{\log z_n}{\log z_{n-1}} - 1\right)\log z_{n-1} + \log(w_n/w_{n+1})},$$

whose limit as $n \to \infty$ is 2 since z_n is an increasing convergent sequence.

This completes the proof of the theorem.

6. NUMERICAL WORK ON THE TRAPEZOID FUNCTION

Indications that the results (2.4) was true were first obtained by MACSYMA [9] computer calculations. It seemed to require 75-digit precision to observe numerically the convergence to 2 of the ratio in the theorem. In this section we give typical results of these calculations. We choose $e = 0.9$. Table II is a list of the results, where

$$s(\lambda_n^e, \lambda_{n-1}^e, \lambda_{n-2}^e, \lambda_{n-3}^e) = \frac{\log\{(\lambda_{n-2}^e - \lambda_{n-3}^e)/(\lambda_{n-1}^e - \lambda_{n-2}^e)\}}{\log\{(\lambda_{n-1}^e - \lambda_{n-2}^e)/(\lambda_n^e - \lambda_{n-1}^e)\}}.$$

TABLE II
The Trapezoid Function with $e = 0.9$

n	$z_n = \lambda_n^e / e$	$s(\lambda_n^e, \lambda_{n-1}^e, \lambda_{n-2}^e, \lambda_{n-3}^e)$
2	1.657	
3	1.709	
4	1.714	
5	1.714	1.89
6	1.714	1.98
7	1.714	1.999 8
8	1.714	1.999 999 97
9	1.714	1.999 999 999 999 999 1

The computations were carried out in 75-digit precision on the MACSYMA computer system [9]. The appropriate roots of the polynomials were found via Newton's method. The Newton method was terminated using the Kantorovich–Akilov criterion, as explained in their book [7 p. 708]. By calculating f_n, f_n' and f_n'', one can determine an interval in which a root of f_n is guaranteed to lie. The width of the interval was set at $2 \cdot 10^{-70}$.

7. Numerical Work on Other Trapezoid-like Functions

We have studied numerically two other trapezoid-like functions.

We define the piecewise linear function for $1/2 < e < 1$ and $1 > c > 0$:

$$
\begin{aligned}
g_1(x, e, c) &= 2cx, && 0 < x \le 1/2,\\
&= \frac{1-c}{e-.5}(x - .5) + c, && 1/2 < x \le e,\\
&= 1, && e < x \le 2 - e,\\
&= \frac{1-c}{.5-c}(x - 1.5) + c, && 2 - c < x \le 1.5,\\
&= -2c(x - 1.5) + c, && 1.5 < x \le 2,
\end{aligned}
$$

and

$$h^1(x, e, c, \lambda) = \lambda g_1(x, e, c).$$

Again write h_p^1 for the pth iterate of h^1 and let $\bar{\lambda}_n$ be the smallest $\lambda > \bar{\lambda}_{n-1}$ which satisfies:

$$h^1_{2^n}(1, e, c, \lambda) = 1$$

TABLE III
The Piecewise Linear Function
with $e = 0.9$, $c = 2/3$

n	$\bar{\lambda}_n$	$s(\bar{\lambda}_n, \bar{\lambda}_{n-1}, \bar{\lambda}_{n-2}, \bar{\lambda}_{n-3})$
1	1.500	
2	1.604	
3	1.640	
4	1.645	1.870
5	1.645	1.986

TABLE IV
The Double Parabola with $e = 0.9$

n	$\bar{\lambda}_n$	$s(\bar{\lambda}_n, \bar{\lambda}_{n-1}, \bar{\lambda}_{n-2}, \bar{\lambda}_{n-3})$
1	1.236	
2	1.379	
3	1.413	
4	1.416	1.870
5	1.416	1.952
6	1.416	1.991

with $\bar{\lambda}_1 > 1$. We then obtain numerically the results given to three digits in Table III.

We have also investigated the function formed from two parabolas, where $0 < e < 1$:

$$\begin{aligned} h^2(x,e) &= \frac{1}{e(2+e)}x(2+x), && 0 \le x \le e, \\ &= 1, && e < x \le 2-e, \\ &= \frac{1}{e(2+e)}(4-x)(2-x), && 2-e < x \le 2. \end{aligned}$$

We define $\bar{\lambda}_n$ as before. The results are given to three digits in Table IV.

8. Numerical Work on Harmonics of Higher Patterns

The theory presented earlier dealt with harmonics of the trapezoid for the pattern R. We then define the nth harmonic of R to be

$$R^{*(n+1)},$$

TABLE V
Harmonics of RL for the Trapezoid Function with $e = 0.9$

n	$z_n = \lambda_n^e/e$	$s(\lambda_n^e, \lambda_{n-1}^e, \lambda_{n-2}^e, \lambda_{n-3}^e)$
2	1.968	
3	1.974	
4	1.974	
5	1.974	1.991
6	1.974	1.999 8
7	1.974	1.999 999 95
8	1.974	1.999 999 999 999 992

using the notation of Derrida *et al.* [3, 4]. We have numerically investigated the harmonics of the pattern RL:

$$RL * R^{*n}$$

for the trapezoid function defined by (2.1). The polynomials analogous to (3.6) for the harmonics of RL are

$$f_1 = -ez^3 + 2z^2 - 1$$

and

$$f_k = -z^{3 \cdot 2^{k-2}} f_{k-1}(z, e) + f_{k-1}(z, 1) \qquad (k > 1).$$

The numerical results are given in Table V to accuracy equal to the number of places given.

9. A Function with All Derivatives Zero at the Peak, but Not Flat

We discuss in this section the numerical behavior of the sequence of λ's associated with period-doubling for the point $x = 1$ of the iterates of the function $F(\lambda, x) = \lambda f(x)$, where

$$f(x) = 2(1 - e^{1-1/(x-1)^2}) \qquad 0 \leq x \leq 2, x \neq 1,$$
$$f(1) = 2,$$

and

$$0 < \lambda < 1.$$

TABLE VI
λ's for Which the Period of $x = 1$ Is 2^n,
Where $F(\lambda, x) = 2(1 - e^{1-1/(x-1)^2})\lambda$

n	λ_n	$(\lambda_{n-1} - \lambda_{n-2})/(\lambda_n - \lambda_{n-1})$
1	0.866585	
2	0.939242	
3	0.945983	10.78
4	0.946422	15.35
5	0.946445	19.10
6	0.946446	22.03
7	0.946446	24.30
8	0.946446	25.68

The function $f(x)$ satisfies:

(a) $f(0) = f(2) = 0$,

(b) $f'(x) < 0$ for $x < 1$,

(c) $f'(x) > 0$ for $x > 1$,

(d) $f^{[k]}(1) = 0$ for each k-derivative.

In these respects $f(x)$ is similar to the flat top functions discussed above. It differs from them in that $f(x)$ is not constant on any interval. If the "squaring" behavior of the λ's for the flat top functions is due to property (d) above, then $\lambda\, f(x)$ should also exhibit this behavior. As we see in Table VI, the numerical evidence is that $F(\lambda, x) = \lambda\, f(x)$ does not have the property of "squaring" for the successive ratios $(\lambda_n - \lambda_{n-1})/(\lambda_{n+1} - \lambda_n)$.

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